A few problems a bit harder than the others

1. Show that for all a, n positive integers:

$$n | \sum_{i=0}^{n-1} a^{(i,n)}$$

where (k, l) denotes the greatest common divisor of k and l.

2. Let n be a positive integer. For $k = 1, 2, \ldots, n$ we define $a_k = \frac{1}{\binom{n}{k}}$ and $b_k = 2^{k-n}$. Show that equality holds:

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \ldots + \frac{a_n - b_n}{n} = 0$$

3. Show that if a, b, c are positive, then an inequality holds:

$$\frac{a}{\sqrt{ab + b^2}} + \frac{b}{\sqrt{bc + c^2}} + \frac{c}{\sqrt{ca + a^2}} \geqslant \frac{3}{\sqrt{2}}$$

4. Given is square chessboard 19×19. Strictly inside this chessboard we specify a smaller square chessboard 17×17. Determine, if the smaller chessboard can be covered using disjoint horizontal rectangles 5×1 and vertical 1×4 placed inside the bigger chessboard.

5. Find all fours of integers x, y, z, t satisfying a system of equations:

$$\begin{cases} xz - 2yt = 3 \\ xt + yz = 1 \end{cases}$$

6. Show, that there exists such an integer k, that $k \cdot 2^n + 1$ is composite for every $n \geqslant 1$.

7. Let $P(x), Q(x), R(x), S(x)$ be such real polynomials, that for every real x equation holds:

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

Show, that $P(1) = 0$.

8. Point P lies inside a triangle ABC. Let D, E, F be perpendicular projections of P onto lines BC, CA, AB. Let O be the circumcentre of triangle DEF, and r be radius of circumcircle of this triangle. Show that:

$$area(ABC) \geqslant 3r\sqrt{3r^2 - 3|OP|^2}$$

9. A circle o is given and two triangles inscribed into it. The common part of them is a hexagon. Show that main diagonals of this hexagon intersect in one point.

10. Median AM of triangle ABC, in which $AB \neq AC$, intersects inscribed circle ω in points K and L. Lines parallel to BC through K and L respectively intersect ω once more in X and Y. Lines AX and AY intersect side BC in points P and Q respectively. Show, that $BP = CQ$.

11. On the plane there is given a disk with radius 1. It has been covered with a square with a side of length 2. The square has been divided into 2007 parallel stripes and one has been thrown away. Show that the disk cannot be covered using remaining stripes.